

# Bianchi I model in terms of nonstandard loop quantum cosmology: Classical dynamics.

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## Abstract

The cosmological singularities of the Bianchi I universe are analyzed in the setting of loop geometry underlying the loop quantum cosmology. We solve the Hamiltonian constraint of the theory and find the Lie algebra of elementary observables. Physical compound observables are defined in terms of elementary ones. Modification of classical theory by holonomy around a loop removes the singularities. However, our model has a free parameter that cannot be determined within our method. Testing the model by the data of observational cosmology may be possible after quantization of our modified classical theory.

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## I. INTRODUCTION

The Bianchi I universe is of primary importance as it underlies, to some extent, the Belinskii-Khalatnikov-Lifshitz (BKL) scenario [1, 2, 3, 4, 5, 6], which is believed to model the Universe in the vicinity of the cosmological singularity. It has been studied recently [8, 9, 10, 11, 12] within *standard* loop quantum cosmology (LQC).

The standard LQC [13, 14] means basically the Dirac method of quantization, which begins with quantization of the *kinematical* phase space followed by imposition of constraints of the gravitational system in the form of operators at the quantum level. Finding kernels of these operators helps to define the *physical* Hilbert space. In the nonstandard LQC [15, 16] one first solves all the constraints at the classical level to identify the *physical* phase space. Next, one identifies the algebra of *elementary* observables (in the physical phase space) and finds its representation. Then, *compound* observables are expressed in terms of elementary ones and quantized. The final goal is finding *spectra* of compound observables which are used to examine the *nature* of the big-bounce phase in the evolution of the Universe.

This paper is devoted to the *classical* dynamics of the Bianchi I model with massless scalar field modified by the loop geometry, described in the framework of the *nonstandard* LQC. The next paper will address the problem of the quantum dynamics [17].

In Sec. II we define the modified classical Hamiltonian. Section III concerns the final choice of canonical variables. The classical dynamics is solved in Sec. IV. An algebra of elementary observables is the subject of Sec. V. Physical observables, that may be confronted with the cosmological data, are defined in Sec. VI. We conclude in the last section. Appendix A presents derivation of the symplectic form on the constraint hypersurface. In Appendix B we derive the algebra of elementary observables corresponding to the Bianchi I model *without* the loop geometry modifications.

## II. HAMILTONIAN

The gravitational part of the classical Hamiltonian,  $H_g$ , in general relativity is a linear combination of the first-class constraints, and reads

$$H_g := \int_{\Sigma} d^3x (N^i C_i + N^a C_a + NC), \quad (1)$$

where  $\Sigma$  is the spacelike part of spacetime  $\mathbb{R} \times \Sigma$ ,  $(N^i, N^a, N)$  denote Lagrange multipliers,  $(C_i, C_a, C)$  are the Gauss, diffeomorphism and scalar constraint functions. In our notation  $(a, b = 1, 2, 3)$  are spatial and  $(i, j, k = 1, 2, 3)$  are internal  $SU(2)$  indices. The constraints must satisfy a specific algebra.

The Bianchi I model with massless scalar field is described by the metric:

$$ds^2 = -N^2 dt^2 + \sum_{i=1}^3 a_i^2(t) dx_i^2, \quad (2)$$

where

$$a_i(\tau) = a_i(0) \left( \frac{\tau}{\tau_0} \right)^{k_i}, \quad d\tau = N dt, \quad \sum_{i=1}^3 k_i = 1 = \sum_{i=1}^3 k_i^2 + k_\phi^2, \quad (3)$$

and where  $k_\phi$  describes matter field density ( $k_\phi = 0$  corresponds to the Kasner model). For clear exposition of the *singularity* aspects of the Bianchi I model we recommend [4, 7].

Having fixed local gauge and diffeomorphism freedom we can rewrite the gravitational part of the classical Hamiltonian, for the Bianchi I model with massless scalar field, in the form [7]

$$H_g = -\gamma^{-2} \int_{\mathcal{V}} d^3x \, N e^{-1} \varepsilon_{ijk} E^{aj} E^{bk} F_{ab}^i, \quad (4)$$

where  $\gamma$  is the Barbero-Immirzi parameter,  $\mathcal{V} \subset \Sigma$  is an elementary cell,  $\Sigma$  is spacelike hypersurface,  $N$  denotes the lapse function,  $\varepsilon_{ijk}$  is the alternating tensor,  $E_i^a$  is a densitized vector field,  $e := \sqrt{|\det E|}$ , and where  $F_{ab}^i$  is the curvature of an  $SU(2)$  connection  $A_a^i$ .

The resolution of the singularity, obtained within LQC, is based on rewriting the curvature  $F_{ab}^k$  in terms of holonomies around loops. The curvature  $F_{ab}^k$  may be determined by making use of the formula

$$F_{ab}^k = -2 \lim_{Ar \square_{ij} \rightarrow 0} Tr \left( \frac{h_{\square_{ij}} - 1}{Ar \square_{ij}} \right) \tau^k \circ \omega_a^i \circ \omega_b^j, \quad (5)$$

where

$$h_{\square_{ij}} = h_i^{(\mu_i)} h_j^{(\mu_j)} (h_i^{(\mu_i)})^{-1} (h_j^{(\mu_j)})^{-1} \quad (6)$$

is the holonomy of the gravitational connection around the square loop  $\square_{ij}$ , considered over a face of the elementary cell, each of whose sides has length  $\mu_j L_j$  (and  $V_o := L_1 L_2 L_3$ ) with respect to the flat fiducial metric  ${}^o q_{ab} := \delta_{ij} {}^o \omega_a^i {}^o \omega_b^j$ ; the fiducial triad  ${}^o e_k^a$  and cotriad  ${}^o \omega_a^k$  satisfy  ${}^o \omega_a^i {}^o e_j^a = \delta_j^i$ ;  $Ar \square_{ij}$  denotes the area of the square; and  $V_o = \int_{\mathcal{V}} \sqrt{{}^o q} d^3x$  is the fiducial volume of  $\mathcal{V}$ . In what follows, to simplify further discussion, we make the assumption  $L_1 = 1 = L_2 = L_3$  (so  $V_o = 1$ ), which is natural in the case of choosing the Bianchi I model with  $T^3$ -topology [11], instead of  $\mathbb{R}^3$ .

The holonomy in the fundamental,  $j = 1/2$ , representation of  $SU(2)$  reads

$$h_i^{(\mu_i)} = \cos(\mu_i c_i / 2) \mathbb{I} + 2 \sin(\mu_i c_i / 2) \tau_i, \quad (7)$$

where  $\tau_i = -i\sigma_i/2$  ( $\sigma_i$  are the Pauli spin matrices). The *connection*  $A_a^k$  and the density weighted *triad*  $E_i^a$  (which occurs in (10)) are determined by the conjugate variables  $c$  and  $p$ :

$$A_a^i = c^i {}^o \omega_a^i, \quad E_i^a = p_i {}^o e_i^a \quad (8)$$

where:

$$c_i = \gamma \dot{a}_i, \quad |p_i| = a_j a_k \quad (9)$$

Making use of (4), (5) and the so-called Thiemann identity

$$\varepsilon_{ijk} e^{-1} E^{aj} E^{bk} = \frac{\text{sgn}(p_1 p_2 p_3)}{2\pi G \gamma (\mu_1 \mu_2 \mu_3)^{1/3}} \sum_k {}^o \varepsilon^{abc} {}^o \omega_c^k Tr \left( h_k^{(\mu_k)} \{ (h_k^{(\mu_k)})^{-1}, V \} \tau_i \right) \quad (10)$$

leads to  $H_g$  in the form

$$H_g = \lim_{\mu_1, \mu_2, \mu_3 \rightarrow 0} H_g^{(\mu_1 \mu_2 \mu_3)}, \quad (11)$$

where

$$H_g^{(\mu_1 \mu_2 \mu_3)} = -\frac{\text{sgn}(p_1 p_2 p_3)}{2\pi G \gamma^3 \mu_1 \mu_2 \mu_3} \sum_{ijk} N \varepsilon^{ijk} Tr \left( h_i^{(\mu_i)} h_j^{(\mu_j)} (h_i^{(\mu_i)})^{-1} (h_j^{(\mu_j)})^{-1} h_k^{(\mu_k)} \{ (h_k^{(\mu_k)})^{-1}, V \} \right), \quad (12)$$

and where  $V = a_1 a_2 a_3$  is the volume of the elementary cell  $\mathcal{V}$ .

The total Hamiltonian for Bianchi I universe with a massless scalar field,  $\phi$ , reads

$$H = H_g + H_\phi \approx 0, \quad (13)$$

where  $H_g$  is defined by (11). The Hamiltonian of the scalar field is known to be:  $H_\phi = N p_\phi^2 |p|^{-3/2} / 2$ , where  $\phi$  and  $p_\phi$  are the elementary variables satisfying  $\{\phi, p_\phi\} = 1$ . The relation  $H \approx 0$  defines the *physical* phase space of considered gravitational system with constraints.

Making use of (7) we calculate (12) and get the *modified* total Hamiltonian  $H_g^{(\lambda)}$  corresponding to (13) in the form

$$H^{(\lambda)} / N = -\frac{1}{8\pi G \gamma^2} \frac{\text{sgn}(p_1 p_2 p_3)}{\mu_1 \mu_2 \mu_3} \left[ \sin(c_1 \mu_1) \sin(c_2 \mu_2) \mu_3 \text{sgn}(p_3) \sqrt{\frac{|p_1 p_2|}{|p_3|}} + \text{cyclic} \right] + \frac{p_\phi^2}{2 \sqrt{V}} \quad (14)$$

where

$$\mu_i := \sqrt{\frac{1}{|p_i|}} \lambda, \quad (15)$$

and where  $\lambda$  is a *regularization* parameter. Here we wish to emphasize that (14) presents a modified *classical* Hamiltonian. It includes *no quantum* physics!

In the gauge  $N = \sqrt{|p_1 p_2 p_3|}$  the Hamiltonian modified by loop geometry reads

$$H^{(\lambda)} = -\frac{1}{8\pi G \gamma^2 \lambda^2} \left[ |p_1 p_2|^{3/2} \sin(c_1 \mu_1) \sin(c_2 \mu_2) + \text{cyclic} \right] + \frac{p_\phi^2}{2}. \quad (16)$$

The Poisson bracket is defined to be

$$\{\cdot, \cdot\} := 8\pi G \gamma \sum_{k=1}^3 \left[ \frac{\partial \cdot}{\partial c_k} \frac{\partial \cdot}{\partial p_k} - \frac{\partial \cdot}{\partial p_k} \frac{\partial \cdot}{\partial c_k} \right] + \frac{\partial \cdot}{\partial \phi} \frac{\partial \cdot}{\partial p_\phi} - \frac{\partial \cdot}{\partial p_\phi} \frac{\partial \cdot}{\partial \phi}, \quad (17)$$

where  $(c_1, c_2, c_3, p_1, p_2, p_3, \phi, p_\phi)$  are canonical variables. The dynamics of  $\xi$  reads

$$\dot{\xi} := \{\xi, H^{(\lambda)}\}, \quad \xi \in \{c_1, c_2, c_3, p_1, p_2, p_3, \phi, p_\phi\}. \quad (18)$$

The dynamics in the *physical* phase space,  $\mathcal{F}_{phys}^{(\lambda)}$ , is defined by solutions to (18) satisfying the condition  $H^{(\lambda)} \approx 0$ . The solutions of (18) ignoring the constraint  $H^{(\lambda)} \approx 0$  are in the *kinematical* phase space,  $\mathcal{F}_{kin}^{(\lambda)}$ .

### III. NEW CANONICAL VARIABLES

We use the following canonical variables

$$\beta_i := \frac{c_i}{\sqrt{|p_i|}}, \quad v_i := |p_i|^{3/2} \text{sgn}(p_i), \quad (19)$$

where  $i = 1, 2, 3$ . They satisfy the algebra

$$\{\beta_i, v_j\} = 12\pi G \gamma \delta_{ij}, \quad (20)$$

where the Poisson bracket reads

$$\{\cdot, \cdot\} = 12\pi G\gamma \sum_{k=1}^3 \left[ \frac{\partial \cdot}{\partial \beta_k} \frac{\partial \cdot}{\partial v_k} - \frac{\partial \cdot}{\partial v_k} \frac{\partial \cdot}{\partial \beta_k} \right] + \frac{\partial \cdot}{\partial \phi} \frac{\partial \cdot}{\partial p_\phi} - \frac{\partial \cdot}{\partial p_\phi} \frac{\partial \cdot}{\partial \phi}. \quad (21)$$

The Hamiltonian in the variables (19) turns out to be

$$H^{(\lambda)} = \frac{p_\phi^2}{2} - \frac{1}{8\pi G\gamma^2} \left( \frac{\sin(\lambda\beta_1) \sin(\lambda\beta_2)}{\lambda^2} v_1 v_2 + \frac{\sin(\lambda\beta_1) \sin(\lambda\beta_3)}{\lambda^2} v_1 v_3 + \frac{\sin(\lambda\beta_2) \sin(\lambda\beta_3)}{\lambda^2} v_2 v_3 \right), \quad (22)$$

where  $\lambda$  parametrizes the holonomy of connection modifying the Bianchi I model.

## IV. DYNAMICS

### A. Equations of motion

The Hamilton equations of motion read<sup>1</sup>

$$\dot{\beta}_i = -18\pi G \frac{\sin(\lambda\beta_i)}{\lambda} (O_j + O_k), \quad (23)$$

$$\dot{v}_i = 18\pi G v_i \cos(\lambda\beta_i) (O_j + O_k), \quad (24)$$

$$\dot{\phi} = p_\phi, \quad (25)$$

$$\dot{p}_\phi = 0, \quad (26)$$

$$H^{(\lambda)} \approx 0, \quad (27)$$

where

$$O_i := \frac{v_i \sin(\lambda\beta_i)}{12\pi G\gamma\lambda}. \quad (28)$$

### B. Solution to equations of motion

Insertion of (24) into (23) gives

$$d\beta_i = -\frac{\tan(\lambda\beta_i)}{\lambda} \frac{dv_i}{v_i}, \quad (29)$$

which leads to

$$v_i \frac{\sin(\lambda\beta_i)}{\lambda} = \text{const} \quad (30)$$

Therefore,  $O_i$  are constants of motion.

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<sup>1</sup> where  $i, j, k = 1, 2, 3$  and  $i \neq j \neq k$

Making use of (25), (24) and  $\cos(\lambda\beta_i) = \sqrt{1 - \sin(\lambda\beta_i)^2}$  gives

$$\int \frac{dv_i}{\sqrt{v_i^2 - (12\pi G\gamma\lambda O_i)^2}} = 18\pi G \int \frac{(O_j + O_k)}{p_\phi} d\phi. \quad (31)$$

Integration of (31) leads to

$$\ln \left| v_i + \sqrt{v_i^2 - (12\pi G\gamma\lambda O_i)^2} \right| = \frac{18\pi G}{p_\phi} (O_j + O_k) (\phi - \phi_i^0). \quad (32)$$

Thus we have

$$\begin{aligned} 2|v_i| = \exp \left( \frac{18\pi G}{p_\phi} (O_j + O_k) (\phi - \phi_i^0) \right) + (12\pi G\gamma\lambda O_i)^2 \times \\ \times \exp \left( - \frac{18\pi G}{p_\phi} (O_j + O_k) (\phi - \phi_i^0) \right), \end{aligned} \quad (33)$$

which may be rewritten as

$$v_i = 12\pi G\gamma\lambda O_i \cosh \left( \frac{18\pi G}{p_\phi} (O_j + O_k) (\phi - \phi_i^0) - \ln |12\pi G\gamma\lambda O_i| \right). \quad (34)$$

It results from the above solutions that for a nonzero value of  $\lambda$  there is no Big Bang type singularity (for any value of  $\phi$ ). The Big Bang is replaced by the Big Bounce. In [7] one considers the so-called *planar collapse*, but we do not consider this issue here as we are mainly concerned with an *initial* type singularity.

Using (34) it is not difficult to get

$$\sin(\lambda\beta_i) = \frac{1}{\cosh \left( \frac{18\pi G}{p_\phi} (O_j + O_k) (\phi - \phi_i^0) - \ln |12\pi G\gamma\lambda O_i| \right)}. \quad (35)$$

Alternatively, one may solve the equation of motion to get  $\beta$ . From (25) and (23) we obtain

$$\int \frac{\lambda d\beta_i}{\sin(\lambda\beta_i)} = -18\pi G \int \frac{(O_j + O_k)}{p_\phi} d\phi. \quad (36)$$

The integration of (36) gives

$$\ln \left| \tan \left( \frac{\lambda\beta_i}{2} \right) \right| = -18\pi G \frac{(O_j + O_k)}{p_\phi} \phi + \text{const.} \quad (37)$$

Removing the cosmological singularities does not complete our task. In what follows we consider an algebra of *elementary* observables and physical *compound* observables as they define the background of the nonstandard LQC [15, 16].

## V. OBSERVABLES

A function  $F$  defined on the phase space is a Dirac observable if it is a solution to the equation

$$\{F, H^{(\lambda)}\} \approx 0. \quad (38)$$

An explicite form of (38) is given by

$$12\pi G\gamma \sum_{i=1}^3 \left( \frac{\partial F}{\partial \beta_i} \frac{\partial H^{(\lambda)}}{\partial v_i} - \frac{\partial F}{\partial v_i} \frac{\partial H^{(\lambda)}}{\partial \beta_i} \right) + \frac{\partial F}{\partial \phi} p_\phi = 0, \quad (39)$$

which due to (22) reads

$$18\pi G \sum_{i=1}^3 \left[ v_i \cos(\lambda \beta_i) \frac{\partial F}{\partial v_i} - \frac{\sin(\lambda \beta_i)}{\lambda} \frac{\partial F}{\partial \beta_i} \right] \cdot (O_j + O_k) + \frac{\partial F}{\partial \phi} p_\phi = 0. \quad (40)$$

### A. Kinematical observables

One may easily verify that  $O_i$  satisfy (40). Instead of solving (40) one may use the constants that occur in (34) and (35). This way we get

$$A_i = \ln \left| \frac{\tan\left(\frac{\lambda \beta_i}{2}\right)}{\frac{\lambda}{2}} \right| + 18\pi G \frac{(O_j + O_k)}{p_\phi} \phi \quad (41)$$

The observables (41) are called *kinematical* as they are not required to satisfy the constraint (27).

### B. Dynamical observables

An explicite form of the constraint (27) in terms of  $O_i$  is given by

$$p_\phi \operatorname{sgn}(p_\phi) = 6\sqrt{\pi G} \sqrt{O_1 O_2 + O_1 O_3 + O_2 O_3}. \quad (42)$$

It results from (22), (27) and (28) that  $O_1 O_2 + O_1 O_3 + O_2 O_3 \geq 0$  so (42) is well defined. Thus, the *dynamical* observables,  $A_i^{\text{dyn}}$ , corresponding to (41) read

$$A_i^{\text{dyn}} = \ln \left| \frac{\tan\left(\frac{\lambda \beta_i}{2}\right)}{\frac{\lambda}{2}} \right| + \frac{3\sqrt{\pi G} \operatorname{sgn}(p_\phi) (O_j + O_k) \phi}{\sqrt{O_1 O_2 + O_1 O_3 + O_2 O_3}}. \quad (43)$$

### C. Algebra of observables

One may verify that  $A_i^{\text{dyn}}$  satisfy the following Lie algebra

$$\{O_i, O_j\} = 0, \quad (44)$$

$$\{A_i^{\text{dyn}}, O_j\} = \delta_{ij}, \quad (45)$$

$$\{A_i^{\text{dyn}}, A_j^{\text{dyn}}\} = 0. \quad (46)$$

In the *physical* phase space the Poisson brackets are found to be (see, Appendix A)

$$\{\cdot, \cdot\}_{\text{dyn}} := \sum_{i=1}^3 \left( \frac{\partial \cdot}{\partial A_i^{\text{dyn}}} \frac{\partial \cdot}{\partial O_i} - \frac{\partial \cdot}{\partial O_i} \frac{\partial \cdot}{\partial A_i^{\text{dyn}}} \right), \quad (47)$$

and the algebra reads

$$\{O_i, O_j\}_{\text{dyn}} = 0, \quad (48)$$

$$\{A_i^{\text{dyn}}, O_j\}_{\text{dyn}} = \delta_{ij}, \quad (49)$$

$$\{A_i^{\text{dyn}}, A_j^{\text{dyn}}\}_{\text{dyn}} = 0. \quad (50)$$

## VI. COMPOUND OBSERVABLES

In what follows we consider the *physical* observables which characterize the singularity aspects of the Bianchi I model. It is helpful to rewrite (42) and (34) in the form

$$p_\phi^2 = 36\pi G (O_1 O_2 + O_1 O_3 + O_2 O_3), \quad (51)$$

$$v_i = 12\pi G \gamma \lambda |O_i| \cosh \left( \frac{3\sqrt{\pi G} \operatorname{sgn}(p_\phi) (O_j + O_k) \phi}{\sqrt{O_1 O_2 + O_1 O_3 + O_2 O_3}} + \ln \left( \frac{\lambda}{2} \right) - A_i^{\text{dyn}} \right). \quad (52)$$

The so-called directional energy density [7] is defined to be

$$\rho_i(\lambda, \phi) := \frac{p_\phi^2}{2 v_i^2}. \quad (53)$$

The *bounce* in the  $i$ -th direction occurs when  $\rho_i$  approaches its maximum [7], which happens at the minimum of  $v_i$  ( $p_\phi$  is a constant of motion). One may easily verify that in the case when all three directions coincide, which corresponds to the Friedmann-Robertson-Walker (FRW) model, these densities turn into the energy density of the flat FRW with massless scalar field [15].

It is clear that  $v_i$  takes minimum for  $\cosh(\cdot) = 1$  so we have

$$v_i^{\min} = 12\pi G \gamma \lambda O_i, \quad \rho_i^{\max} = \frac{1}{2} \left( \frac{p_\phi}{12\pi G \gamma \lambda O_i} \right)^2. \quad (54)$$

Rewriting  $O_i$  and  $p_\phi$  in terms of  $k_i$  and  $k_\phi$  [7]

$$O_i = \frac{2}{3} k_i K, \quad p_\phi = \sqrt{8\pi G} k_\phi K, \quad (55)$$

where  $K$  is a constant, leads to

$$\rho_i^{\max} = \frac{1}{16\pi G \gamma^2 \lambda^2} \left( \frac{k_\phi}{k_i} \right)^2. \quad (56)$$

We can determine  $\rho_i^{\max}$  if we know  $\lambda$ . However,  $\lambda$  is a free parameter of the formalism. Thus, finding the critical energy densities of matter corresponding to the cosmic singularities of the Bianchi I model is an open problem.



One may apply (56) to the Planck scale. Substituting  $\lambda = l_{Pl}$  gives

$$\rho_i^{max} \simeq 0,35 \left( \frac{k_\phi}{k_i} \right)^2 \rho_{Pl}, \quad (57)$$

which demonstrates that  $\rho_i^{max}$  may fit the Planck scale depending on the ratio  $k_\phi/k_i$ .

Another important physical observable is the *volume* of the Universe. From the definitions (9) and (19) we get

$$V = a_1 a_2 a_3 = |v_1 v_2 v_3|^{1/3}. \quad (58)$$

It results from (52), (55) and (3) that the volume is *bounded* from below.

## VII. CONCLUSIONS

The *modification* of the classical Hamiltonian by using loop geometry turns classical singularities of the Bianchi I model into Big Bounces, similarly as in the case of the initial singularity of the FRW type models [15].

Our approach is quite different from the so-called effective or polymerization method (see, e.g. [7]), where the replacement  $\beta \rightarrow \sin(\lambda\beta)/\lambda$  in the Hamiltonian finishes the procedure of quantization. In our method this replacement has been done entirely at the *classical* level. Quantization consists in finding a self-adjoint representation of observables on the *physical* phase space and an examination of the spectra of these observables [15, 16].

The *elementary* observables constitute a complete set of constants of motion on the constraint surface. They are used to parametrize the physical phase space and are “building blocks” for the *compound* observables like the directional energy density and the volume operator. So they have deep physical meaning. Their role becomes even more important at the *quantum* level as they enable finding quantum operators corresponding to the classical compound observables [16].

The modification of classical theory by holonomy of connection around a loop *does not* change the Lie algebra of elementary observables. The algebras of modified (48)-(50) and nonmodified (B9)-(B11) observables are, to some extent, isomorphic. It is a valuable feature of the modification procedure.

Our quantum Bianchi I model has a *free* parameter  $\lambda$  that cannot be determined within the model. This parameter, similarly to the case of the FRW [15], is expected to be fixed by the *data* of observational cosmology (see, e.g. [18]).

The algebra of elementary observables is defined on the *physical* phase space. The compound observables are thus defined on the physical phase space too. Thus, their properties may be confronted with the data of observational cosmology, i.e. *real* world.

The maximum of the energy densities may fit the Planck scale which seems to be surprising as the equation used concerns only the classical level. But a similar situation occurred in the case of the FRW model [15] and the explanation has come from the examination of the quantum energy density operator. Its *spectrum* turned out to coincide with the classical counterpart [16].

The present paper gives a background for *quantization* of the Bianchi I model [17]. The quantization is required despite the fact that the *singularity* problem is resolved already at the classical level due to the modifications based on the loop geometry via holonomy of connection. It is so because the *spectra* of the quantum observables may be used to get a

link with the data of observational cosmology, similarly as in the case of the FRW model [19, 20]. Another reason is that the Big Bounces may occur at any energy density (being parametrized by a free parameter) so it is necessary to quantize the model.

Our paper shows that the *nonstandard* LQC method, worked out for the case of the FRW model [15], may be applied to the Bianchi I model. It seems to be applicable to the isotropic models (e.g. Lemaître) as well.

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## APPENDIX A: SYMPLECTIC FORM

The symplectic form on the physical phase space,  $\Omega$ , may be obtained from the symplectic form on the kinematical phase space,  $\omega$ , by taking into account the constraint (42).

The symplectic form corresponding to (21) reads

$$\omega = \frac{1}{12\pi G\gamma} \sum_{i=1}^3 d\beta_i \wedge dv_i + d\phi \wedge dp_\phi. \quad (\text{A1})$$

Making use of the constraint (42) gives

$$dp_\phi = \sum_{i=1}^3 \left( \frac{\partial p_\phi}{\partial \beta_i} d\beta_i + \frac{\partial p_\phi}{\partial v_i} dv_i \right) = \sum_{i=1}^3 \frac{3\sqrt{\pi G} \operatorname{sgn}(p_\phi)(O_j + O_k)}{\sqrt{O_1 O_2 + O_1 O_3 + O_2 O_3}} dO_i. \quad (\text{A2})$$

Thus, we have

$$d\phi \wedge dp_\phi = \sum_{i=1}^3 \frac{3\sqrt{\pi G} \operatorname{sgn}(p_\phi)(O_j + O_k)}{\sqrt{O_1 O_2 + O_1 O_3 + O_2 O_3}} d\phi \wedge dO_i = \sum_{i=1}^3 \frac{\partial A_i^{\text{dyn}}}{\partial \phi} d\phi \wedge dO_i. \quad (\text{A3})$$

On the other hand (28) leads to

$$dv_i = \frac{12\pi G\gamma\lambda}{\sin(\lambda\beta_i)} dO_i - \cot(\lambda\beta_i) v_i d\beta_i, \quad (\text{A4})$$

which gives

$$d\beta_i \wedge dv_i = \frac{12\pi G\gamma\lambda}{\sin(\lambda\beta_i)} d\beta_i \wedge dO_i. \quad (\text{A5})$$

Thus, we have

$$\frac{1}{12\pi G\gamma} \sum_{i=1}^3 d\beta_i \wedge dv_i = \sum_{i=1}^3 \frac{\lambda}{\sin(\lambda\beta_i)} d\beta_i \wedge dO_i. \quad (\text{A6})$$

It results from (A3) and (A5) that we have

$$\Omega = \sum_{i=1}^3 \left( \frac{\lambda}{\sin(\lambda\beta_i)} d\beta_i + \frac{\partial A_i^{\text{dyn}}}{\partial \phi} d\phi \right) \wedge dO_i. \quad (\text{A7})$$

Now, let us rewrite  $dA_i^{\text{dyn}}$  as follows

$$dA_i^{\text{dyn}} = \frac{\partial A_i^{\text{dyn}}}{\partial \beta_i} d\beta_i + \frac{\partial A_i^{\text{dyn}}}{\partial v_i} dv_i + \sum_{j \neq i} \frac{\partial A_i^{\text{dyn}}}{\partial \beta_j} d\beta_j + \sum_{j \neq i} \frac{\partial A_i^{\text{dyn}}}{\partial v_j} dv_j + \frac{\partial A_i^{\text{dyn}}}{\partial \phi_i} d\phi_i. \quad (\text{A8})$$

Next, we make summation

$$\begin{aligned} \sum_{i=1}^3 dA_i^{\text{dyn}} = \sum_{i=1}^3 \left( \frac{\lambda}{\sin(\lambda\beta_i)} d\beta_i + \frac{3\sqrt{\pi G} \text{sgn}(p_\phi) (\mathcal{O}_j + \mathcal{O}_k)^2}{(\mathcal{O}_1\mathcal{O}_2 + \mathcal{O}_1\mathcal{O}_3 + \mathcal{O}_2\mathcal{O}_3)^{3/2}} d\mathcal{O}_i + \frac{\partial A_i^{\text{dyn}}}{\partial \phi_i} d\phi_i + \right. \\ \left. + \sum_{j \neq i} \frac{\partial A_i^{\text{dyn}}}{\partial \beta_j} d\beta_j + \sum_{j \neq i} \frac{\partial A_i^{\text{dyn}}}{\partial v_j} dv_j \right). \end{aligned} \quad (\text{A9})$$

Eventually, we calculate the wedge product with  $d\mathcal{O}_i$

$$\sum_{i=1}^3 dA_i^{\text{dyn}} \wedge d\mathcal{O}_i = \sum_{i=1}^3 \left( \frac{\lambda}{\sin(\lambda\beta_i)} d\beta_i + \frac{\partial A_i^{\text{dyn}}}{\partial \phi_i} d\phi_i \right) \wedge d\mathcal{O}_i. \quad (\text{A10})$$

Finally, the physical symplectic form  $\Omega$  reads

$$\Omega = \sum_{i=1}^3 dA_i^{\text{dyn}} \wedge d\mathcal{O}_i. \quad (\text{A11})$$

Thus, the physical phase space may be parametrized by the variables  $A_i^{\text{dyn}}$  and  $\mathcal{O}_i$ , and the corresponding Poisson bracket is given by (47).

## APPENDIX B: NONMODIFIED CASE

To find an algebra of elementary observables, for this case, we introduce the definitions

$$\mathcal{O}_i := \lim_{\lambda \rightarrow 0} \mathcal{O}_i = \frac{v_i \beta_i}{12\pi G \gamma}, \quad (\text{B1})$$

$$\mathcal{A}_i := \lim_{\lambda \rightarrow 0} A_i = \ln |\beta_i| + 18\pi G \frac{(\mathcal{O}_j + \mathcal{O}_k)}{p_\phi} \phi. \quad (\text{B2})$$

In the limit  $\lambda \rightarrow 0$  the Hamiltonian constraint (51) turns into “unmodified” constraint

$$p_\phi \text{sgn}(p_\phi) = 6\sqrt{\pi G} \sqrt{\mathcal{O}_1\mathcal{O}_2 + \mathcal{O}_1\mathcal{O}_3 + \mathcal{O}_2\mathcal{O}_3}. \quad (\text{B3})$$

It results from (B1)-(B3) that

$$\mathcal{A}_i^{\text{dyn}} = \ln |\beta_i| + \frac{3\sqrt{\pi G} \text{sgn}(p_\phi) (\mathcal{O}_j + \mathcal{O}_k) \phi}{\sqrt{\mathcal{O}_1\mathcal{O}_2 + \mathcal{O}_1\mathcal{O}_3 + \mathcal{O}_2\mathcal{O}_3}}. \quad (\text{B4})$$

One may verify that

$$\{\mathcal{O}_i, \mathcal{O}_j\} = 0, \quad (\text{B5})$$

$$\{\mathcal{A}_i^{\text{dyn}}, \mathcal{O}_j\} = \delta_{ij}, \quad (\text{B6})$$

$$\{\mathcal{A}_i^{\text{dyn}}, \mathcal{A}_j^{\text{dyn}}\} = 0, \quad (\text{B7})$$

and

$$\Omega = \sum_{i=1}^3 d\mathcal{A}_i^{\text{dyn}} \wedge d\mathcal{O}_i. \quad (\text{B8})$$

Direct calculations lead to

$$\{\mathcal{O}_i, \mathcal{O}_j\}_{\text{dyn}} = 0, \quad (\text{B9})$$

$$\{\mathcal{A}_i^{\text{dyn}}, \mathcal{O}_j\}_{\text{dyn}} = \delta_{ij}, \quad (\text{B10})$$

$$\{\mathcal{A}_i^{\text{dyn}}, \mathcal{A}_j^{\text{dyn}}\}_{\text{dyn}} = 0, \quad (\text{B11})$$

which coincides with the algebraic structure of the algebra of the modified observables (48)-(50).

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